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Generalized matrix Riccati equations with superposition formulae

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Abstract

Ordinary differential equations with superposition formulae related to parabolic subgroups of $SL(n)$ are explicitly found. It is shown that these equations can be reduced to a system of matrix Riccati equations.

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Introduction

A system of ordinary differential equations (ODEs) has a superposition formula if it is possible to express any solution as a function of some particular solutions and arbitrary constants. More than 100 years ago Lie proved that such systems of equations correspond to finite-dimensional subalgebras in the algebra of vector fields in $\mathbb{C}^N$ or $\mathbb{R}^N$. Thus, an ODE with superposition formula can be associated with any action of a Lie group G on a manifold M.

The ODEs with superposition formulae appear in soliton theory as Bäcklund transformations, usually (but not always) as matrix Riccati equations. Matrix Riccati equations occur in many other applications, e.g. in optimal control theory, diffusion problems, etc [9]. Thus, we may hope to find applications of our generalized matrix Riccati equation in these areas.

We consider the case when $G = SL(n)$ and M is a homogeneous space $SL(n)/P$, where P is a parabolic subgroup of $SL(n)$. The papers [1–3] deal with the case of a primitive action of Lie groups, which corresponds to maximal parabolic subgroups. Here we consider the case of a non-primitive action. This case was investigated for some particular choices of P in the paper [4].

Our goal is to construct explicitly the corresponding ODE for an arbitrary parabolic subgroup P and show that the solving of this equation can be reduced to the solving of a matrix Riccati equation.

1. Ordinary differential equations with superposition formulae

Let us consider a first-order ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{y} = f(\vec{y}, t) \tag{1}$$

where $\vec{y} \in \mathbb{C}^N$ or $\mathbb{R}^N$. We shall say that this equation has a superposition formula if the general solution $\vec{y}(t)$ can be expressed as a function of a finite number m of particular solutions and N free constants

$$\vec{y}(t) = F(\vec{y}_1(t), \ldots, \vec{y}_m, c_1, \ldots, c_N) \tag{2}$$

where $\vec{y}_i(t)$, $i = 1, \ldots, m$, are particular solutions of (1) and c_i, $i = 1, \ldots, N$, are arbitrary constants.

Lie [5] proved the following theorem.

Theorem 1 (Lie). *Equation (1) allows a superposition formula if and only if the function f has the form*

$$f(\vec{y}, t) = \sum_{k=1}^{r} Z_k(t)\vec{\xi}_k(\vec{y})$$

where the vector functions $\vec{\xi}_k(\vec{y})$ are such that vector fields

$$X_k = \sum_{\mu=1}^{N} \xi_k^{\mu}(\vec{y})\partial_{y^\mu}$$

generate a finite-dimensional Lie subalgebra $\mathfrak{h}$ of the algebra of vector fields on $\mathbb{C}^N$ or $\mathbb{R}^N$, i.e.

$$[X_k, X_l] = \sum_{j=1}^{r} C_{klj} X_j.$$

If we have an action of a Lie group G on a manifold M then we have an induced homomorphism from an Lie algebra $\mathfrak{g}$ to an algebra of vector fields on M. Thus, in this situation we obtain an ODE with superposition formulae.

Let us consider the case when $G = SL(n)$ and M is a homogeneous space $SL(n)/P$, where P is any parabolic subgroup of $SL(n)$. We recall that a parabolic subgroup is any subgroup containing the Borel subgroup, i.e. the maximal solvable subgroup (the Borel subgroup of a semisimple complex Lie group is unique, up to conjugacy).

Our goal is to construct explicitly the corresponding ODE and show that the solving of this equation can be reduced to the solving of a matrix Riccati equation.

2. Explicit construction of the ODE

A parabolic subgroup $P \subset SL(n)$ corresponds to an ordered set $n_1, \ldots, n_k$ of fixed positive integers such that $n = \sum_{i=1}^{k} n_i$, since the subgroup P can always be realized by matrices of the form

$$\begin{pmatrix} P_{11} & P_{12} & \cdots & P_{1k} \\ 0 & P_{22} & \cdots & P_{2k} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & P_{kk} \end{pmatrix}$$

where each P_{ij} is a matrix $n_i \times n_j$.

Consider a homogeneous space $SL(n)/P$ and the canonical action of $SL(n)$ on $SL(n)/P$

$$g_1 \cdot gP = g_1 gP.$$

We have to find a suitable coordinate system on the space $SL(n)/P$. We will construct natural coordinates in the biggest Schubert cell.

Consider a set $M \subset SL(n)$ of matrices

$$\begin{pmatrix} A_{11} & \cdots & A_{1k} \\ \vdots & \ddots & \vdots \\ A_{k1} & \cdots & A_{kk} \end{pmatrix}$$

such that A_{ij} is an $n_i \times n_j$ matrix and

$$|A_{11}| \neq 0 \qquad \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} \neq 0, \ldots, \begin{vmatrix} A_{11} & \cdots & A_{1k} \\ \vdots & \ddots & \vdots \\ A_{k1} & \cdots & A_{kk} \end{vmatrix} \neq 0. \tag{3}$$

Lemma 1. *For any $A \in M$ there exists a unique $B \in P$ such that AB has the form*

$$\begin{pmatrix} I_1 & 0 & \cdots & 0 \\ & I_2 & \cdots & 0 \\ & & \ddots & \vdots \\ * & & & I_k \end{pmatrix}$$

where I_i is the $n_i \times n_i$ identity matrix.

Proof. The proof is an elementary corollary of the conditions (3); it can be found in [6]. □

One can easily verify that for any $B \in P$

$$M \cdot B = M.$$

Therefore M is a union of cosets of P. It follows from lemma 1 that in each coset in M there is a unique element of the form

$$\begin{pmatrix} I_1 & 0 & \cdots & 0 \\ K_{21} & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ K_{k1} & \cdots & K_{k,k-1} & I_k \end{pmatrix} \tag{4}$$

where K_{ij} is an $n_i \times n_j$ matrix. It is well known that the image $p(M)$ by the canonical projection $p : SL(n) \longrightarrow SL(n)/P$ is diffeomorphic to a Euclidean space (this is a so-called generalized Schubert cell; see, e.g., [7]).

As coordinates of a point $x \in p(M)$ we can choose elements of the matrices $K_{21}, K_{31}, K_{32}, \ldots, K_{k,k-1}$ of the unique matrix K of the form (4) in the coset of P corresponding to the point $x \in p(M)$.

We shall say that a matrix K of the form (4) is a coordinate matrix. By $K(x)$ we denote a coordinate matrix corresponding to $x \in p(M)$. In fact, the coordinates of x are the elements of K_{ij} $(i > j \geqslant 1)$.

Now let us consider a group action. Consider a point $x \in SL(n)/P$ and the corresponding coordinates $K(x)$. After multiplication by $g \in SL(n)$ we obtain $gK(x)$. But this matrix is not, in general, a matrix of the form (4), so it is necessary to multiply $gK(x)$ on the right by an appropriate matrix $B \in P$. Thus, $K(gx) = gK(x)B$, where B depends on both g and $K(x)$. This procedure is quite implicit. Nevertheless, we can obtain an explicit answer (this is done in [6]), but really we do not need it since we are interested in the corresponding homomorphism of algebras. Let us find this homomorphism.

Consider a path $g(t) \in SL(n)$ such that $g(0) = e$, $\frac{\mathrm{d}}{\mathrm{d}t} g(t)|_{t=0} = X$ for some $X \in \mathfrak{sl}(n)$. Thus, for an element X of $\mathfrak{sl}(n)$ we obtain a vector field $\tilde{X} = \frac{\mathrm{d}}{\mathrm{d}t} K(gx)|_{t=0}$ (we use identification of a vector space and its tangent space in a point to shorten our notation). Thus, we obtain a vector field

$$\tilde{X}(x) = \frac{\mathrm{d}}{\mathrm{d}t} K(gx)\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} g(t)\bigg|_{t=0} (K(x)B(t))|_{t=0} + (g(t)K(x))|_{t=0} \frac{\mathrm{d}}{\mathrm{d}t} B(t)\bigg|_{t=0}$$
$$= XK(x) + K(x)B'$$

where $B' = \frac{\mathrm{d}}{\mathrm{d}t} B(t)|_{t=0}$. The matrix B was defined by the condition that $gK(x)B$ has the form (4), so B' is defined by the condition that $\tilde{X}(x) = XK(x) + K(x)B'$ has the form

$$\begin{pmatrix} 0 & \cdots & 0 \\ & \ddots & \vdots \\ * & & 0 \end{pmatrix} \tag{5}$$

where a 0 in the position i, j is an $n_i \times n_j$ matrix.

After this we can construct our ODE following a general procedure. Thus, for a path $X(t)$ in a Lie algebra $\mathfrak{sl}(n)$ we obtain an ODE with superposition formula

$$\frac{\mathrm{d}}{\mathrm{d}t} K(x) = \tilde{X}(x, t).$$

We can find this ODE explicitly. For convenience we will split the matrix $K(x)$ into its submatrices as in (4) and we will split the matrix $X \in \mathfrak{sl}(n)$ into its submatrices

$$X = \begin{pmatrix} X_{11} & \cdots & X_{1,k} \\ \vdots & & \vdots \\ X_{k1} & \cdots & X_{kk} \end{pmatrix}$$

where X_{ij} is an $n_i \times n_j$ matrix.

Theorem 2. *The ODE corresponding to a homogeneous space $SL(n)/P$ is given by the following formula:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} K_{i+1,i} \\ \cdots \\ K_{k,i} \end{pmatrix} = \begin{pmatrix} X_{i+1,i} \\ \cdots \\ X_{k,i} \end{pmatrix} + \begin{pmatrix} X_{i+1,i+1} & \cdots & X_{i+1,k} \\ \vdots & & \vdots \\ X_{k,i+1} & \cdots & X_{k,k} \end{pmatrix} \begin{pmatrix} K_{i+1,i} \\ \cdots \\ K_{k,i} \end{pmatrix}$$
$$- \begin{pmatrix} K_{i+1,1} & \cdots & K_{i+1,i} \\ \vdots & & \vdots \\ K_{k,1} & \cdots & K_{k,i} \end{pmatrix} \left[\sum_{\beta=0}^{i-1} (-1)^\beta \begin{pmatrix} 0 & 0 & \cdots & 0 \\ K_{21} & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ K_{i1} & \cdots & K_{i,i-1} & 0 \end{pmatrix}^\beta \right]$$
$$\times \left[\begin{pmatrix} X_{1,i} \\ \cdots \\ X_{i,i} \end{pmatrix} + \begin{pmatrix} X_{1,i+1} & \cdots & X_{1,k} \\ \vdots & & \vdots \\ X_{i,i+1} & \cdots & X_{i,k} \end{pmatrix} \begin{pmatrix} K_{i+1,i} \\ \cdots \\ K_{k,i} \end{pmatrix} \right] \tag{6}$$

where $i = 1, \ldots, k-1$. We use the standard convention that, for a square matrix A, zero power A^0 is equal to the identity matrix.

Proof. From the formula $\tilde{X} = XK(x) + K(x)B'$ we obtain the following formula:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} K_{i+1,i} \\ \cdots \\ K_{k,i} \end{pmatrix} = \begin{pmatrix} X_{i+1,i} \\ \cdots \\ X_{k,i} \end{pmatrix} + \begin{pmatrix} X_{i+1,i+1} & \cdots & X_{i+1,k} \\ \vdots & & \vdots \\ X_{k,i+1} & \cdots & X_{k,k} \end{pmatrix}\begin{pmatrix} K_{i+1,i} \\ \cdots \\ K_{k,i} \end{pmatrix} + \begin{pmatrix} K_{i+1,1} & \cdots & K_{i+1,i} \\ \vdots & & \vdots \\ K_{k,1} & \cdots & K_{k,i} \end{pmatrix}\begin{pmatrix} B'_{1i} \\ \vdots \\ B'_{ii} \end{pmatrix} \tag{7}$$

where the B'_{ij} are submatrices of B', defined in the same way as the submatrices X_{ij} of X.

Next, let us find the matrices B'_{ij}. They are defined by the condition that $\tilde{X}$ has the form (5). It is easy to rewrite this condition as a system of equations

$$\begin{pmatrix} X_{1i} \\ \vdots \\ X_{ii} \end{pmatrix} + \begin{pmatrix} X_{1,i+1} & \cdots & X_{1k} \\ \vdots & & \vdots \\ X_{i,i+1} & \cdots & X_{i,k} \end{pmatrix}\begin{pmatrix} K_{i+1,i} \\ \vdots \\ K_{k,i} \end{pmatrix} + \begin{pmatrix} I & 0 & \cdots & 0 \\ K_{21} & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ K_{i1} & \cdots & K_{i,i-1} & I \end{pmatrix}\begin{pmatrix} B'_{i1} \\ \vdots \\ B'_{ii} \end{pmatrix} = 0$$

where $i = 1, \ldots, k$. It is well known that if A is a nilpotent matrix then

$$I + \sum_{i=1}^{\infty}(-1)^i A^i$$

is the inverse matrix for $I + A$. Thus, we can solve the system of equations for B'_{ij} and we obtain the formulae

$$\begin{pmatrix} B'_{i1} \\ \vdots \\ B'_{ii} \end{pmatrix} = -\left[\sum_{\beta=0}^{i-1}(-1)^{\beta}\begin{pmatrix} 0 & 0 & \cdots & 0 \\ K_{21} & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ K_{i1} & \cdots & K_{i,i-1} & 0 \end{pmatrix}^{\beta}\right] \times \left[\begin{pmatrix} X_{1,i} \\ \cdots \\ X_{i,i} \end{pmatrix} + \begin{pmatrix} X_{1,i+1} & \cdots & X_{1,k} \\ \vdots & & \vdots \\ X_{i,i+1} & \cdots & X_{i,k} \end{pmatrix}\begin{pmatrix} K_{i+1,i} \\ \cdots \\ K_{k,i} \end{pmatrix}\right].$$

It is sufficient to substitute these formulae in (7) to prove the theorem. □

3. Relation with the matrix Riccati equation

Let us recall that a matrix Riccati equation is an equation of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}W(t) = A(t) + B(t)W(t) + W(t)C(t) + W(t)D(t)W(t) \tag{8}$$

where A, B, C, D and W are matrices of appropriate sizes. The matrix Riccati equation is an equation with a superposition formula; it was studied in the papers [3, 8], where the superposition formulae were found.

We now want to prove the conjecture from the paper [6] that we can represent our ODE related to $SL(n)/P$ as a list of differential equations such that the first equation is a matrix Riccati equation and the others are also matrix Riccati equations but with coefficients depending on the solutions of the previous equations.

We will prove this conjecture using theorem 2. Indeed, the equation (6) for $i = 1$ is a matrix Riccati equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} K_{21} \\ \cdots \\ K_{k1} \end{pmatrix} = \begin{pmatrix} X_{21} \\ \cdots \\ X_{k1} \end{pmatrix} + \begin{pmatrix} X_{22} & \cdots & X_{2k} \\ \vdots & & \vdots \\ X_{k2} & \cdots & X_{k,k} \end{pmatrix}\begin{pmatrix} K_{21} \\ \cdots \\ K_{k1} \end{pmatrix} - \begin{pmatrix} K_{21} \\ \vdots \\ K_{k1} \end{pmatrix}(X_{11}) - \begin{pmatrix} K_{21} \\ \vdots \\ K_{k1} \end{pmatrix}(X_{12} \quad \cdots \quad X_{1k})\begin{pmatrix} K_{21} \\ \cdots \\ K_{k1} \end{pmatrix}.$$

Let us now consider the equation (6) for $i > 1$. We can rewrite it in the following form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} K_{i+1,i} \\ \cdots \\ K_{k,i} \end{pmatrix} = A + B\begin{pmatrix} K_{i+1,i} \\ \cdots \\ K_{k,i} \end{pmatrix} + \begin{pmatrix} K_{i+1,i} \\ \cdots \\ K_{k,i} \end{pmatrix} C + \begin{pmatrix} K_{i+1,i} \\ \cdots \\ K_{k,i} \end{pmatrix} D \begin{pmatrix} K_{i+1,i} \\ \cdots \\ K_{k,i} \end{pmatrix} \tag{9}$$

where

$$A = \begin{pmatrix} X_{i+1,i} \\ \cdots \\ X_{k,i} \end{pmatrix} - \begin{pmatrix} K_{i+1,1} & \cdots & K_{i+1,i-1} \\ \vdots & & \vdots \\ K_{k,1} & \cdots & K_{k,i-1} \end{pmatrix} Q_1 \begin{pmatrix} X_{1,i} \\ \cdots \\ X_{i,i} \end{pmatrix}$$

$$B = \begin{pmatrix} X_{i+1,i+1} & \cdots & X_{i+1,k} \\ \vdots & & \vdots \\ X_{k,i+1} & \cdots & X_{k,k} \end{pmatrix} - \begin{pmatrix} K_{i+1,1} & \cdots & K_{i+1,i-1} \\ \vdots & & \vdots \\ K_{k,1} & \cdots & K_{k,i-1} \end{pmatrix} Q_1 \begin{pmatrix} X_{1,i+1} & \cdots & X_{1k} \\ \vdots & & \vdots \\ X_{i,i+1} & \cdots & X_{ik} \end{pmatrix}$$

$$C = -Q_2 \begin{pmatrix} X_{1i} \\ \cdots \\ X_{ii} \end{pmatrix} \qquad D = -Q_2 \begin{pmatrix} X_{1,i+1} & \cdots & X_{1k} \\ \vdots & & \vdots \\ X_{i,i+1} & \cdots & X_{ik} \end{pmatrix}$$

$$Q_1 = \sum_{\beta=0}^{i-2} (-1)^\beta \begin{pmatrix} 0 & 0 & \cdots & 0 \\ K_{21} & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ K_{i-1,1} & \cdots & K_{i-1,i-2} & 0 \end{pmatrix}^\beta$$

$$Q_2 = \sum_{\beta=0}^{i-1} (-1)^\beta \begin{pmatrix} 0 & 0 & \cdots & 0 \\ K_{21} & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ K_{i1} & \cdots & K_{i,i-1} & 0 \end{pmatrix}^\beta .$$

We see that equation (9) is a matrix Riccati equation (8) for

$$W = \begin{pmatrix} K_{i+1,i} \\ \cdots \\ K_{ki} \end{pmatrix}$$

with coefficients A, B, C, D depending on K_{mn} with $n < i$, which ends the proof of our conjecture.

It should be remarked that the initial equation was related to $SL(n)$, but the obtained matrix Riccati equations are generally related to $GL(n)$.

Thus, we obtain that the problem of solving the ODE (6) can be reduced to the problem of solving a sequence of matrix Riccati equations. We can also reduce the problem of finding an explicit superposition formula for the equation (6) to the problem of finding an explicit superposition formula for a matrix Riccati equation.

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